

NECESSARY CONDITIONS FOR FRACTIONAL HARDY-SOBOLEV'S INEQUALITIES

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ABSTRACT.

In this short article we obtain some necessary conditions for a so-called fractional Hardy-Sobolev's inequalities in multidimensional case.

We also give some examples to show the sharpness of these inequalities.

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1. INTRODUCTION. STATEMENT OF PROBLEM. NOTATIONS.

A. Ordinary Hardy-Sobolev's fractional inequalities.

The following assertion is called Hardy-Sobolev's (ordinary) difference inequality:

$$\left[\int_D |u(x)|^q |x|^{-\mu} dx \right]^{1/q} \leq K_{HS}(p, q) \times \left[\int_D \int_D \frac{|u(x) - u(y)|^p |x|^{-\alpha(1)} |y|^{-\alpha(2)}}{|x - y|^\beta} dx dy \right]^{1/p}. \quad (1.1a)$$

We will write further $\mu = \lambda dq$.

Here D is open domain with positive Lebesgue measure $\mu(D) = \int_D dx$ in the whole Euclidean space R^d ; $d = 1, 2, \dots$ equipped with ordinary Euclidean norm $|x|$; $x \in R^d$, for instance, whole space R^d or its half space or unit ball, $u = u(\cdot)$ is an arbitrary function from the class $C_0^\infty(D)$, $\alpha(1), \alpha(2), \beta, \lambda = \text{const}$, $p, q = \text{const} \in (1, \infty)$, the finite positive (if there exists) "constant" $K_{HS}(p, q) = K_{HS}(p, q; \alpha(1), \alpha(2); d)$ dependent of the $p, q; \alpha(1), \alpha(2), \beta, \mu; d$ but not of the function $u(\cdot)$.

By means of approximation we can assume $u \in W^{\mu/q, q}$, especially when we investigate the lower bounds for the constants.

The finiteness of integrals in the left-hand and right-hand sizes in (1.1a) for arbitrary function $u(\cdot) \in C_0^\infty(D)$ entrusts the following conditions on the constants $\alpha(1), \alpha(2), \beta, \mu$:

$$\mu < d, \alpha(1) > -d, \alpha(2) > -d, \beta < 1, \alpha(1) + \alpha(2) - \beta > -d. \quad (Ca)$$

We assume also that

$$\lambda \in (0, 1/(2d - 1)). \quad (Cb)$$

We will suppose hereafter the conditions (Ca) and (Cb) are satisfied.

The following generalization of inequality (1.1a) is called Hardy-Sobolev's *weight* difference inequality:

$$\left[\int_D |u(x)|^q W_{-\mu}(x) dx \right]^{1/q} \leq K_{HS}(p, q; W) \times \left[\int_D \int_D \frac{|u(x) - u(y)|^p W_{\alpha}(x, y)}{W_{\beta}(|x - y|)} dx dy \right]^{1/p}. \quad (1.1b)$$

More general case appears if we write instead $u(x)$ the function $u(x) - u(0)$, for instance

$$\left[\int_D |u(x) - u(0)|^q |x|^{-\mu} dx \right]^{1/q} \leq K_{HS}^{(0)}(p, q) \times \left[\int_D \int_D \frac{|u(x) - u(y)|^p |x|^{-\alpha(1)} |y|^{-\alpha(2)}}{|x - y|^{\beta}} dx dy \right]^{1/p}. \quad (1.1c)$$

B. Mixed Hardy-Sobolev's fractional inequalities.

The inequality of a view

$$\left[\int_D |u(x)|^r |x|^{-\mu} dx \right]^{1/r} \leq K_{M;HS}(p, q, r) \times \left\{ \int_D |y|^{\alpha(2)} dy \left[\int_D \frac{|u(x) - u(y)|^p |x|^{\alpha(1)} dx}{|x - y|^{\beta}} \right]^{q/p} \right\}^{1/q} \quad (1.2a)$$

is said to be (ordinary) Mixed Hardy-Sobolev's fractional inequality.

The *weight* version of Mixed Hardy-Sobolev's fractional inequality has a view

$$\left[\int_D |u(x)|^r W_{-\mu}(x) dx \right]^{1/r} \leq K_{MHS;W}(p, q, r) \times \left\{ \int_D dy \left[\int_D \frac{|u(x) - u(y)|^p W_{\alpha}(x, y) dx}{W_{\beta}(|x - y|)} \right]^{q/p} \right\}^{1/q}. \quad (1.2b)$$

The functions $W_{\alpha}(\cdot), W_{\beta}(\cdot), W_{-\mu}(\cdot)$ are *weight* function, i.e. are measurable positive almost everywhere functions.

C. Hardy-Sobolev's fractional derivative-difference inequalities.

By definition, the inequality of a view

$$\left[\int_D \int_D \frac{|u(x) - u(y)|^p |x|^{-\alpha(1)} |y|^{-\alpha(2)}}{|x - y|^{\beta}} dx dy \right]^{1/p} \leq K_{DD;HS}(p, s) \times \int_D [|\nabla u(x)|^s |x|^{-\mu} dx]^{1/s} \quad (1.3c)$$

is called fractional derivative-difference inequality.

Here for the vector $x = \{x_1, x_2, \dots, x_d\}$

$$\nabla u(x) = \text{grad } u \stackrel{\text{def}}{=} (\partial u / \partial x_1, \partial u / \partial x_2, \dots, \partial u / \partial x_d).$$

We omit here and in the next pilcrow the obvious weight generalization of these inequalities.

D. Surface Hardy-Sobolev's fractional inequalities.

The following assertion is named as surface Hardy-Sobolev's difference inequality:

$$\left[\int_S |u(x)|^q |x|^{-\mu} \sigma(dx) \right]^{1/q} \leq K_{S;HS}(p, q) \times \left[\int_D \int_D \frac{|u(x) - u(y)|^p |x|^{-\alpha(1)} |y|^{-\alpha(2)}}{|x - y|^\beta} dx dy \right]^{1/p}. \quad (1.4a)$$

Here S is smooth (sub)surface of a boundary ∂D of a dimensional m ; $1 \leq m \leq d - 1$ with correspondent surface measure $d\sigma(x) = \sigma(dx)$.

Our aim is finding of some necessary conditions for the fractional Hardy-Sobolev's inequalities.

We obtain also the lower bounds for the constants $K_{HS}(p, q)$, $K_{M;HS}(p, q, r)$, $K_{S;HS}(p, q)$, $K_{DD;HS}(p, s)$ and consider some generalizations on the so-called Grand Lebesgue spaces instead classical Lebesgue-Riesz's L_p spaces.

Some upper estimations for Hardy-Sobolev's fractional inequalities see, e.g. in the works [1], [5], [6],[7], [2],[3], [4], [30],[11], [12], [13], [14], [15], [16] etc.

The one-dimensional case $d = 1$ for the ordinary Hardy-Sobolev's inequality was investigated before by Jakovlev [9] and Grisvard [8].

About applications of these inequalities see, e.g. [3],[4], [7],[12], [16].

We use the symbols $C(X, Y)$, $C(p, q; \psi)$, etc., to denote positive constants along with parameters they depend on, or at least dependence on which is essential in our study. To distinguish between two different constants depending on the same parameters we will additionally enumerate them, like $C_1(X, Y)$ and $C_2(X, Y)$. The relation $g(\cdot) \asymp h(\cdot)$, $p \in (A, B)$, where $g = g(p)$, $h = h(p)$, $g, h : (A, B) \rightarrow R_+$, denotes as usually

$$0 < \inf_{p \in (A, B)} h(p)/g(p) \leq \sup_{p \in (A, B)} h(p)/g(p) < \infty.$$

The symbol \sim will denote usual equivalence in the limit sense.

We will denote as ordinary the indicator function

$$I(x \in A) = 1, x \in A, \quad I(x \in A) = 0, x \notin A;$$

here A is a measurable set.

2. MAIN RESULT: NECESSARY CONDITIONS FOR THE HARDY-SOBOLEV'S FRACTIONAL INEQUALITIES

A. Ordinary Hardy-Sobolev's fractional inequalities.

Theorem 1A. Let in the inequality (1.1a) $D = R^d$ and suppose (1.1a) be satisfied for some non-constant function $u(\cdot) \in C_0^\infty$. Then

$$\frac{d - \mu}{q} = \frac{2d + \alpha(1) + \alpha(2) - \beta}{p}. \quad (2.1a)$$

Proof used the so-called dilation method belonging to G.Talenty [35]. Namely, let $\theta = \text{const} \in (0, \infty)$. The dilation operator T_θ may be defined as follows:

$$u_\theta(x) = T_\theta u(x) \stackrel{\text{def}}{=} u(x/\theta).$$

Note that if $u(\cdot) \in C_0^\infty$, then $u_\theta(\cdot) \in C_0^\infty$.

We obtain substituting the function $u_\theta(\cdot)$ into inequality (1.1a) instead the function $u(\cdot)$ after changing variables $x = \theta z, y = \theta v$:

$$\begin{aligned} & \theta^{(d-\mu)/q} \left[\int_D |u(x) - u(0)|^q |x|^{-\mu} dx \right]^{1/q} \leq K_{HS}(p, q) \times \\ & \theta^{(2d+\alpha(1)+\alpha(2)-\beta)/p} \left[\int_D \int_D \frac{|u(x) - u(y)|^p |x|^{-\alpha(1)} |y|^{-\alpha(2)}}{|x - y|^\beta} dx dy \right]^{1/p}. \end{aligned}$$

Since the value θ is arbitrary in the set $(0, \infty)$, we conclude

$$(d - \mu)/q = (2d + \alpha(1) + \alpha(2) - \beta)/p,$$

Q.E.D.

Analogously may be proved the following results of this section.

B. Mixed Hardy-Sobolev's fractional inequalities.

Theorem 1B. Let in the inequality (1.2a) $D = R^d$ and suppose (1.2a) be satisfied for some non-constant function $u(\cdot) \in C_0^\infty$. Then

$$\frac{d - \mu}{r} = \frac{d + \alpha(2) - \beta}{p} + \frac{d + \alpha(1)}{q}. \quad (2.1b)$$

C. Hardy-Sobolev's fractional difference-derivative inequalities

Theorem 1C. Let in the inequality (1.3a) $D = R^d$ and suppose (1.3a) be satisfied for some non-constant function $u(\cdot) \in C_0^\infty$. Then

$$\frac{d - \mu}{s} - 1 = \frac{2d + \alpha(1) + \alpha(2) - \beta}{p}. \quad (2.1c)$$

D. Hardy-Sobolev's surface fractional difference-derivative inequalities

We consider here the case when

$$x \in S \Leftrightarrow x_{m+1} = x_{m+2} = \dots = x_n = 0.$$

We obtain using as before at the same dilation method as when in the proof of inequality (1.4a) holds then

$$\frac{m - \mu}{q} = \frac{2d + \alpha(1) + \alpha(2) - \beta}{p}. \quad (2.1d)$$

3. WEIGHT GENERALIZATIONS OF HARDY-SOBOLEV'S FRACTIONAL INEQUALITIES.

We consider in this section the Hardy-Sobolev's *weight* difference inequality in the whole space $D = R^d$:

$$\left[\int_{R^d} |u(x)|^q W_{-\mu}(x) dx \right]^{1/q} \leq K_{HS}(p, q; W) \times \left[\int_{R^d} \int_{R^d} \frac{|u(x) - u(y)|^p W_\alpha(x, y)}{W_\beta(|x - y|)} dx dy \right]^{1/p}. \quad (3.1)$$

A new notations. For some finite constant $\alpha_0, \beta_0, \mu_0; \alpha_\infty, \beta_\infty, \mu_\infty$ we introduce a functions, to be presumed non-zero and integrable:

$$\begin{aligned} \underline{W}_{-\mu,0}(z) &= \inf_{\theta \in (0,1)} \frac{W_\mu(\theta z)}{\theta^{-\mu_0}}, \\ \underline{W}_{\beta,0}(z) &= \inf_{\theta \in (0,1)} \frac{W_\beta(\theta z)}{\theta^{\beta_0}}, \\ \overline{W}_{\alpha,0}(z, v) &= \sup_{\theta \in (0,1)} \frac{W_\alpha(\theta z, \theta v)}{\theta^{\alpha_0}}; \\ \underline{W}_{-\mu,\infty}(z) &= \inf_{\theta \in (1,\infty)} \frac{W_\mu(\theta z)}{\theta^{-\mu_\infty}}, \\ \underline{W}_{\beta,\infty}(z) &= \inf_{\theta \in (1,\infty)} \frac{W_\beta(\theta z)}{\theta^{\beta_\infty}}, \\ \overline{W}_{\alpha,\infty}(z, v) &= \sup_{\theta \in (1,\infty)} \frac{W_\alpha(\theta z, \theta v)}{\theta^{\alpha_\infty}}. \end{aligned}$$

Theorem 3.1. If the inequality (3.1) is satisfied for any non-zero function $u \in C_0^\infty(R^d)$, then

$$\frac{d - \mu_0}{q} \geq \frac{d + \alpha_0 - \beta_0}{p}, \quad (3.2a)$$

$$\frac{d - \mu_\infty}{q} \leq \frac{d + \alpha_0 - \beta_0}{p}, \quad (3.2b)$$

As a **corollary**: when $\mu_0 = \mu_\infty = \mu, \alpha_0 = \alpha_\infty = \alpha, \beta_0 = \beta_\infty = \beta$, then both the inequalities (3.2a) and (3.2b) reduced to the known relation

$$\frac{d - \mu}{q} = \frac{d + \alpha - \beta}{p}. \quad (3.2c)$$

Proof is alike to the proof of theorem 2.1; it used the Talenty dilation method and splitting into two cases: $\theta \in (0, 1)$ and $\theta \in (1, \infty)$.

4. LOWER BOUNDS FOR CONSTANTS IN HARDY-SOBOLEV'S FRACTIONAL INEQUALITIES. EXAMPLES.

We denote by $\overline{K}_{HS}, \overline{K}_{M;HS}, \overline{K}_{S;HS}, \overline{K}_{DD;HS}$ the *minimal values* of the constants $K_{HS} = K_{HS}(p, q), K_{M;HS} = K_{M;HS}(p, q, r), K_{S;HS} = K_{S;HS}(p, q), K_{DD;HS} = K_{DD;HS}(p, s)$ in the (correspondingly) Hardy-Sobolev's, Mixed Hardy-Sobolev's, Surface Hardy-Sobolev's, and Differential - Difference Hardy-Sobolev's inequalities for the whole space $D = R^d$.

Note that for some particular cases of domains D (half-spaces etc.) the exact values of these constants was calculated by K.Bogdan and B.Dyda [1], R.Frank, R.Seiringer [2].

For instance, $\overline{K}_{HS} = \overline{K}_{HS}(p, q) = \overline{K}_{HS}(p, q; \alpha(1), \alpha(2), \lambda) =$

$$\sup_{u \in C_0^\infty(R^d)} \left\{ \left[\int_D |u(x)|^q |x|^{-\mu} dx \right]^{1/q} : \left[\int_D \int_D \frac{|u(x) - u(y)|^p |x|^{-\alpha(1)} |y|^{-\alpha(2)}}{|x - y|^\beta} dx dy \right]^{1/p} \right\}. \quad (4.0)$$

Recall that $\mu = \lambda dp, \lambda \in (0, 1)$.

Theorem 4.a. Suppose the domain D contains some ball

$$B(t) = \{x : |x| \leq t\}, \quad B = B(1)$$

with the center in origin of the whole space R^d .

Let also in the ordinary Hardy-Sobolev's inequality $1 \leq p < d/\lambda$ and $\alpha(1) = \alpha(2) = 0, \beta = d(1 + \lambda p)$. Assume that the condition (2.1a) is satisfied. Then

$$C_1(\lambda; d) \left[\frac{p}{|p - d/\lambda|} \right]^\lambda \leq \overline{K}_{HS}(p, p; 0, 0, \lambda) \leq C_2(\lambda; d) \cdot \left[\frac{p}{|p - d/\lambda|} \right]. \quad (4.1a)$$

Proof. The upper estimation contains in fact in [1]; see also [2]; see also [8], [9], [11].

Without loss of generality we can assume $d = 1$; the multidimensional case $d \geq 2$ is investigated analogously, with at the same (counter)example.

It is sufficient to consider the one-dimensional case $d = 1$ and to obtain the lower estimate to consider the following example in the case when the domain D contains the unit ball of the set R^d , with the center in origin:

$$u_0(x) = |\log |x|| \cdot I(|x| \leq 1).$$

We have consequently as $p \in [1, 1/\lambda), p \rightarrow 1/\lambda - 0$:

$$L^p \stackrel{\text{def}}{=} 2 \int_0^1 x^{-\lambda p} |\log x| dx = \frac{\Gamma(p+1)}{|1 - \lambda p|^{p+1}};$$

$$L \asymp |p - 1/\lambda|^{1+1/p} \asymp |p - 1/\lambda|^{1+\lambda}.$$

Further,

$$R^p := \left[\int_D \int_D \frac{[|\log |x| - \log |y||]^p}{|x - y|^\beta} dx dy \right] \asymp \left[\int_B \frac{[|\log |x| - \log |y||]^p}{|x - y|^\beta} dx dy \right] \asymp$$

$$\int_0^1 \rho^{d-1-\beta} |\log \rho| d\rho \times \int_0^{2\pi} \frac{|\log |\tan \phi||^p}{|\cos \phi - \sin \phi|^\beta} d\phi =: I_1 \cdot I_2.$$

Note that the second integral I_2 is bounded when $1 \leq p < 1/\lambda$ and

$$I_1 \sim \frac{C}{|1 - \lambda p|}.$$

This completes the proof of theorem 4.a.

Remark 4.1a. We obtain in more general case when $\alpha(1) \geq 0, \alpha(2) \geq 0, \alpha := \alpha(1) + \alpha(2) > 0, \mu = \alpha - \lambda d p$ denoting

$$p_0 = \frac{1}{\lambda} + \frac{\alpha}{\lambda d} :$$

$$C_1(\alpha(1), \alpha(2), \lambda; d) \left[\frac{p}{|p - p_0|} \right]^{1/p_0} \leq \overline{K}_{HS}(p, p; \alpha(1), \alpha(2), \lambda) \leq$$

$$C_2(\alpha(1), \alpha(2), \lambda; d) \cdot \left[\frac{p}{|p - p_0|} \right].$$

Analogously may be obtained the following results.

Theorem 4.b. Let in the mixed ordinary Hardy-Sobolev's inequality $1 \leq r < d/\lambda$ and $\beta = d(1 + \lambda p)$. Assume the condition (2.1b) is satisfied. Then

$$C_3(\alpha(1), \alpha(2), \lambda; d) \left[\frac{1}{|r - d/\lambda|} \right]^\lambda \leq \overline{K}_{M;HS}(p, q) \leq$$

$$C_4(\alpha(1), \alpha(2), \lambda; d) \cdot \left[\frac{1}{|r - d/\lambda|} \right]. \quad (4.1b)$$

Theorem 4.c. Let in the differential-difference Hardy-Sobolev's inequality $\beta = d(1 + \lambda p)$ and let the condition (2.1c) be satisfied. Then

$$C_5(\alpha(1), \alpha(2), \lambda; d) \leq \overline{K}_{DD;HS}(p, s) \leq C_6(\alpha(1), \alpha(2), \lambda; d). \quad (4.1c)$$

Theorem 4.d. Let in the surface Hardy-Sobolev's inequality $1 \leq q < m/\lambda$ and $\beta = d(1 + \lambda p)$. Assume also the condition (2.1d) is satisfied. Then

$$C_7(\alpha(1), \alpha(2), \lambda; d, m) \left[\frac{q}{|q - m/\lambda|} \right]^\lambda \leq \overline{K}_{S;HS}(p, q) \leq$$

$$C_8(\alpha(1), \alpha(2), \lambda; d, m) \cdot \left[\frac{q}{|q - m/\lambda|} \right]. \quad (4.1d)$$

5. GENERALIZATION ON A BILATERAL GRAND LEBESGUE SPACES

We recall briefly the definition and needed properties of these spaces. More details see in the works [22], [23], [25], [26], [33], [34], [29], [27], [28] etc.

For a and b constants, $1 \leq a < b \leq \infty$, let $\psi = \psi(p)$, $p \in (a, b)$, be a continuous positive function such that there exists a limits (finite or not) $\psi(a+0)$ and $\psi(b-0)$, with conditions $\inf_{p \in (a,b)} \psi > 0$ and $\min\{\psi(a+0), \psi(b-0)\} > 0$. We will denote the set of all these functions as $\Psi(a, b)$.

The Bilateral Grand Lebesgue Space (in notation BGLS) $G(\psi; a, b) = G(\psi)$ is the space of all measurable functions $f : D \rightarrow R$ or $f : R^d \rightarrow R$ endowed with the norm

$$\|f\|_{G(\psi)} \stackrel{def}{=} \sup_{p \in (a,b)} \left[\frac{|f|_p}{\psi(p)} \right], \quad (5.1)$$

if it is finite.

The $G(\psi)$ spaces over some measurable space (X, F, μ) with condition $\mu(X) = 1$ (probabilistic case) appeared in an article [29].

The BGLS spaces are rearrangement invariant spaces and moreover interpolation spaces between the spaces $L_1(R^d)$ and $L_\infty(R^d)$ under real interpolation method [?], [27].

It was proved also that in this case each $G(\psi)$ space coincides with the so - called *exponential Orlicz space*, up to norm equivalence. In others quoted publications were investigated, for instance, their associate spaces, fundamental functions $\phi(G(\psi; a, b); \delta)$, Fourier and singular operators, conditions for convergence and compactness, reflexivity and separability, martingales in these spaces, etc.

Let $g : X \rightarrow R$ be some measurable function such that $\exists(a, b), 1 \leq a < b \leq \infty$, such that $\forall p \in (a, b) \Rightarrow |g|_p < \infty$.

We can then introduce the non-trivial function $\psi_g(p)$ as follows:

$$\psi_g(p) \stackrel{def}{=} |g|_p, \quad p \in (a, b). \quad (5.2)$$

This choosing of the function $\psi_g(\cdot)$ will be called *natural choosing*.

Remark 1. If we introduce the *discontinuous* function

$$\psi_r(p) = 1, \quad p = r; \quad \psi_r(p) = \infty, \quad p \neq r, \quad p, r \in (a, b) \quad (5.3)$$

and define formally $C/\infty = 0$, $C = \text{const} \in R^1$, then the norm in the space $G(\psi_r)$ coincides with the L_r norm:

$$\|f\|_{G(\psi_r)} = |f|_r.$$

Thus, the Bilateral Grand Lebesgue spaces are the direct generalization of the classical exponential Orlicz's spaces and Lebesgue spaces L_r .

The BGLS norm estimates, in particular, Orlicz norm estimates for measurable functions, e.g., for random variables are used in PDE [22], [25], theory of probability in Banach spaces [31], [29], [33], in the modern non-parametrical statistics, for example, in the so-called regression problem [33].

Let us introduce the following linear operators acting on the function $u(\cdot) \in C_0^\infty(R^d)$:

$$\delta_\lambda[u](x, y) = \frac{u(x) - u(y)}{|x - y|^{\lambda d}}, \quad (5.4)$$

$$S_\lambda[u](x) = \frac{u(x)}{|x|^{\lambda d}}. \quad (5.5)$$

Let also $\psi_2 = \psi_2(p)$ be some function from the class $\Psi(a, b; R^d \times R^d)$ relative a new *potential* measure

$$\nu(dx, dy) = \frac{dxdy}{|x - y|^{\lambda d}} \quad (5.6)$$

and such that $b \leq 1/\lambda$ or conversely $a \geq 1/\lambda$.

Put

$$\psi_1(p) = \overline{K}_{HS}(p) \cdot \psi_2(p).$$

Theorem 5.1a.

$$\|S_\lambda[u]\|G\psi_2 \leq 1 \cdot \|\delta_\lambda[u]\| |x - y|^{-\lambda d} \|G\psi_1, \quad (5.7)$$

where the constant "1" is the best possible.

Proof is very simple. Let

$$\delta_\lambda[u] |x - y|^{-\lambda d} \in G\psi_2;$$

without loss of generality we can suppose $\|\delta_\lambda[u]\| |x - y|^{-\lambda d} \|G\psi_2 = 1$.

From the direct definition of the norm in Grand Lebesgue spaces it follows

$$\left[\int_{R^d} \int_{R^d} \frac{|\delta_\lambda[u]|^p(x, y)}{|x - y|^{\lambda d}} dxdy \right]^{1/p} \leq \psi_2(p), \quad p \in (a, b).$$

We obtain using the Hardy-Sobolev's inequality with $q = p$:

$$\left[\int_{R^d} |S_\lambda[u]|^p(x) dx \right]^{1/p} \leq \overline{K}_{HS}(p) \cdot \psi_2(p) = \psi_1(p),$$

which is equivalent to the assertion of our theorem.

The precision of the constant "1" follows immediately from the main result of paper [17].

Note that it follows from upper estimation for the constant $K_{HS}(p)$ that if we define a new function $\psi_3(p)$ as follows:

$$\psi_3(p) = \frac{p \psi_2(p)}{|1/\lambda - p|},$$

then

$$\|S_\lambda[u]\|G\psi_3 \leq C \cdot \|\delta_\lambda[u]\|G\psi_1.$$

This result is weakly exact in the following sense. Let $\psi_4(p)$ be each function from the class $G\Psi(a, b)$, where either $a = 1/\lambda$ or $b = 1/\lambda$ for which

$$\lim_{p \rightarrow 1/\lambda} \frac{p \psi_4(p)}{|p - 1/\lambda|^\lambda} = 0.$$

Then for the function $u_0(x) = |\log |x|| \cdot I(|x| \leq 1)$

$$\lim_{p \rightarrow 1/\lambda} \frac{||S_\lambda[u_0]||G\psi_4}{||\delta_\lambda[u_0]||G\psi_1} = \infty.$$

The *mixed, or anisotropic Grand Lebesgue Spaces* was introduced in [18]. Indeed, let $u = u(x)$, $x \in R^n$ be measurable function: $u : R^n \rightarrow R$. Recall that the anisotropic Lebesgue space $L_{\vec{p}}$ consists on all the functions f with finite norm

$$|f|_{\vec{p}} \stackrel{def}{=} \left(\int_{R_1^{m_1}} \mu_1(dx_1) \left(\int_{R_2^{m_2}} \mu_2(dx_2) \dots \left(\int_{R_l^{m_l}} |f(\vec{x})|^{p_1} \mu_l(dx_l) \right)^{p_2/p_1} \right)^{p_3/p_2} \dots \right)^{1/p_l}. \quad (5.8)$$

Here $m_j = \dim x_j$, $\sum_j m_j = n$.

Note that in general case

$$|f|_{p_1, p_2} \neq |f|_{p_2, p_1},$$

but

$$|f|_{p, p} = |f|_p.$$

Observe also that if $f(x_1, x_2) = g_1(x_1) \cdot g_2(x_2)$ (condition of factorization), then

$$|f|_{p_1, p_2} = |g_1|_{p_1} \cdot |g_2|_{p_2},$$

(formula of factorization).

Let $\nu = \nu(\vec{p})$ be some continuous positive on the set Q ; $\vec{p} \in Q$ function such that

$$\inf_{p \in Q} \nu(p) > 0, \quad \nu(p) = \infty, \quad p \notin Q. \quad (5.9)$$

We denote the set all of such a functions as Ψ_Q .

The (multidimensional, anisotropic) Grand Lebesgue Spaces $GLS = G_Q(\nu) = G_Q\nu$ space consists on all the measurable functions $f : R^n \rightarrow R$ with finite norms

$$||f||G_Q(\nu) \stackrel{def}{=} \sup_{\vec{p} \in Q} [|f|_{\vec{p}} / \nu \vec{p}]. \quad (5.10)$$

In the further considered case $n = 2d$, $\mu_1(dx) = \mu_2(dx) = dx$.

Theorem 5.2. Let for some $\nu \in G\nu$

$$\frac{\delta_\lambda(x, y)}{|x - y|^d} \in G\nu(R^d \times R^d).$$

Define the domain R_r , $r \geq 1$ (sub-domain in the plane R^2) by the following way:

$$R_r = R_r(\alpha(1), \alpha(2), \beta, \mu; d) = \{(p, q) : p, q \geq 1, \frac{d - \mu}{r} = \frac{d + \alpha(2) - \beta}{p} + \frac{d + \alpha(1)}{q}\} \quad (5.11)$$

and the function

$$\psi_5(r) = \inf_{(p, q) \in R_r} [\nu(p, q) K_{M; HS}(p, q)]. \quad (5.12)$$

Assertion:

$$\|S_\lambda u\|_{G\psi_5} \leq \|\delta_\lambda u \cdot |x - y|^{-\lambda d}\|_{G\nu}. \quad (5.13)$$

Proof. Without loss of generality we can and do suppose $\|f\|_{G_Q(\nu)} = 1$; then

$$\|\delta_\lambda u\|_{p,q} \leq \nu(p, q).$$

It follows from the definition of the norm in BGLS spaces and the mixed norm inequality (2.1b) that

$$|S_\lambda u|_r \leq \nu(p, q) K_{M;HS}(p, q), \quad (p, q) \in R_r, \quad (5.14)$$

therefore

$$|S_\lambda u|_r \leq \inf_{(p,q) \in R_r} [\nu(p, q) K_{M;HS}(p, q)] = \psi_5(r). \quad (5.15)$$

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